Section 1: Interest rate and time value of money

Outline

- Interest rate
- Type of interest: Compound and Simple interest rate
- Nominal rate of interest
- Discount Rate or Rate of Discount
- Nominal Rate of discount
- Continuous compound rate
- Effective rate with changing duration
- Force of interest
- Present Value and Future Value
- Accumulation factor and Discount factor

Interest rate

Interest rate

Symbol: i

Unit: Percent (usually per annum)

Sample: Saving account offer an interest rate of 0.5% per annum

Amount of Interest

Symbol: I

Unit: Currency (THB, USD, CNY, etc.)

Sample: I will receive $800 as an interest payment

Formulas

Let

$$A[t] = \text{Amount of money in the account at time } t$$

1. $A[t] = A[t-1] + I$

2. $I = i * A[t-1]$

3. $A[t] = A[t-1] + i * A[t-1]$

4. $i = \frac{A[t]-A[t-1]}{A[t-1]}$

Sample Question

1.Let A[0]=100 and A[1]=102.Find I and i in year 1.

2. Let $A[t] = t^2 + 100$. Find I and i in year 2.

3. Let $A[t] = 100 * 1.06^t$.Find I and i in year 2 and 3.

Type of Interest

- Simple Interest
- Compound interest or effective rate of interest

Simple Interest

- "I" is constant every year
- Only the principle at time 0 earn interest.

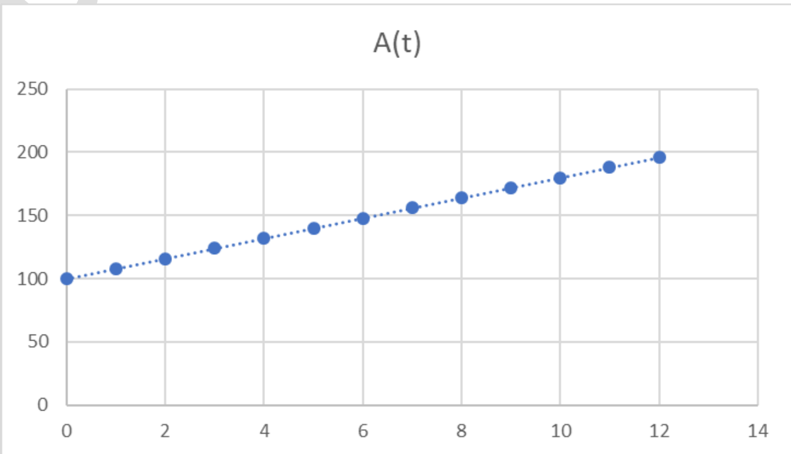
Formula

$$A[t] = A[0] + I_s * A[0] * t$$

- $I_s \neq i$

Year[t]	-2	-1	0	1	2
General	$A[0] - 2 * I * A[0]$	$A[0] - 1 * I * A[0]$	A[0]	$A[0] + 1 * I * A[0]$	$A[0] + 2 * I * A[0]$
A[0]=100, I=8%	84	92	100	108	116
A[0]=1, I=0.02	0.96	0.98	1	1.02	1.04

Linear function



Sample questions

1. Account A[t] earns Simple Interest at $I_s = 8\%$ and A[0]=100. Find A[1], A[2],A[8],A[9], i_1,i_2 and i_9

2. Account A[t] earns Simple Interest at $I_s = 5\%$ and A[0]=100. Find i_1,i_2,i_3 and i_4

Compound interest or effective rate of interest

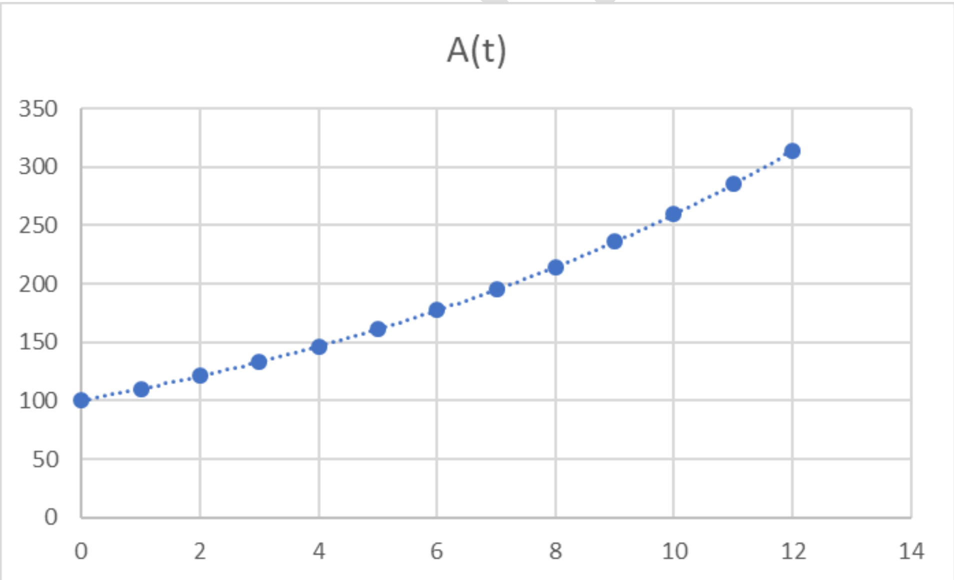
- Both principle and interest earn interest
- "i" is constant

Formula

$$A[t] = A[0] * (1 + i)^t$$

Year[t]	-2	-1	0	1	2
General	$\frac{A[0]}{(1+i)^2}$	$\frac{A[0]}{(1+i)^1}$	A[0]	$A[0] * (1+i)^1$	$A[0] * (1+i)^2$
A[0]=100, I=8%	85.73388203	92.59259259	100	108	116.64
A[0]=1, I=0.02	0.961168781	0.980392157	1	1.02	1.0404

Exponential function



Sample question

1.Let A[t] receive a compound interest rate with A[0]=100 and i = 7%. Find A[1], A[2], A[3] และ i_1, i_2, i_3

Nominal Rate of interest

- Nominal: Just in name
- Nominal rate differs slightly from the real rate

Sample Question

1. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of 2% compound/convertible/pays semiannually. Find the effective[real] rate of interest for 1 year

$$i^{[2]} = Nominal\ rate\ of\ interest\ compound\ semiannually[6month]$$

$$i^{[2]} = 2\%$$

2. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of 12% compound/convertible/pays monthly. Find the effective[real] rate of interest for 1 year

$$i^{[12]} = Nominal\ rate\ of\ interest\ compound\ monthly[1month]$$

$$i^{[12]} = 12\%$$

3. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of 3% compound/convertible/pays every 2 years. Find the effective[real] rate of interest for 1 year

$$i^{[1/2]} = Nominal\ rate\ of\ interest\ compound\ every\ 2\ years$$

$$i^{[1/2]} = 3\%$$

Formula

$$(1+\frac{i^{[m]}}{m})^m = 1 + i$$

Discount Rate or Rate of Discount

- Similar to Interest Rate
- Interest → to the future
- Discount → to the past
- Know one, know the other
- Symbol: d

$$i = \frac{A[t] - A[t-1]}{A[t-1]}$$

$$d = \frac{A[t] - A[t-1]}{A[t]}$$

Year[t]	-2	-1	0	1	2
General[interest]	$\frac{A[0]}{(1+i)^2}$	$\frac{A[0]}{(1+i)^1}$	A[0]	$A[0]*(1+i)^1$	$A[0]*(1+i)^2$
General[Discount]	$A[0]*(1-d)^2$	$A[0]*(1-d)^1$	A[0]	$\frac{A[0]}{(1-d)^1}$	$\frac{A[0]}{(1-d)^2}$

Year	-2	-1	0	1	2
A[0]=100, I=3%			100		
A[0]=100, D=3%			100		

Sample Question

1. Fund A gain interest at the effective rate of interest of 3%. Find the equivalent discount rate

2. Fund B gains interest at the nominal rate 4% compound quarterly. Find the equivalent Discount rate.

3. Fund C gains a 4% discount rate. Find the equivalent interest rate.

Formula

$$A[0] * (1+i)^1 = \frac{A[0]}{(1-d)^1}$$

$$1 + i = \frac{1}{1-d}$$

Nominal Rate of discount

$(1+\frac{i^{[m]}}{m})^m = 1+i$ → Interest

$(1-\frac{d^{[m]}}{m})^m = 1-d$ → Discount

- Similar to Nominal Rate of interest
- Symbol: $d^{(M)}$

Sample Question

1. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of discount of 2% compound/convertible/pays semiannually. Find the effective[real] rate of interest and effective[real] rate of discount for 1 year

$$d^{[2]} = Nominal\ rate\ of\ discount\ compound\ semiannually[6month]$$

$$d^{[2]} = 2\%$$

2. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of discount of 12% compound/convertible/pays monthly. Find the effective[real] rate of interest and effective[real] rate of discount for 1 year

$$d^{[12]} = Nominal\ rate\ of\ interest\ compound\ monthly[1 month]$$

$$d^{[12]} = 12\%$$

3. Actuarial tutor bank pays Interest for their fixed deposit account at nominal rate of discount at 3% compound/convertible/pays every 2 years. Find the effective[real] rate of interest for 1 year

$$d^{[1/2]} = Nominal\ rate\ of\ interest\ compound\ every\ 2\ years$$

$$d^{[1/2]} = 3\%$$

4. A[t] receive interest at a nominal rate of discount of 12% compound monthly. Find the equivalent rate of interest

5. A[t] receive interest at a nominal rate of discount of 6% compound semiannually. Find the equivalent rate of discount

Formula

$$(1-\frac{d^{[m]}}{m})^m = 1-d$$

Continuous compound rate

Pays interest continuously

$$i^{(M)}\ as\ M \to \infty$$

Formula

$$\lim_{m\to\infty}(1+\frac{I^{[m]}}{m})^m = e^{i^{(\infty)}} = 1+i$$

Proof

Sample Question

1. Given the continuous compound interest rate of 6%. How much would 100 grows to
   In 6 month
   In 1 Year
   In 3 Years

2. Determine the effective[real] rate of interest for any 6 month period, Given the continuous compound interest rate is 10%

3. Given a continuous compound rate $I^{(\infty)}$ and initial fund of A(0), calculate A(t)

Equivalent rate: know one, know the rest

Concept:

Equivalent rate → If initial principals are equal, then money at time 1 is equal

Rate	Time -1	Time 0	Time 1
Interest rate	$\frac{1}{1+i}$	1	$1+i$
Discount rate	$1-d$	1	$\frac{1}{1-d}$
Nominal interest	$\frac{1}{(1+\frac{i^{(m)}}{m})^m}$	1	$(1+\frac{i^{(m)}}{m})^m$
Nominal discount	$(1-\frac{d^{(m)}}{m})^m$	1	$\frac{1}{(1-\frac{d^{(m)}}{m})^m}$
Continuous compound	$e^{-i^{(\infty)}}$	1	$e^{i^{(\infty)}}$

If the rates are equal, the following equation are true:

Effective rate with changing duration

Concept:

Equivalent rate → If initial principals are equal, then money at time 1 is equal

Example: If the effective rate per year is 3%, calculate the effective rate per 6 months

Example: If the effective rate per year is 6%, calculate the effective rate per 3 months

Example: If the effective rate per year is 10%, calculate the effective rate per 2 years

Example: If the nominal rate per year compound monthly is 12%, calculate the effective rate per 1 month, 2 months and 1 year

Example: If the discount rate per year is 3%, calculate the effective rate per 1 months and 6 months.

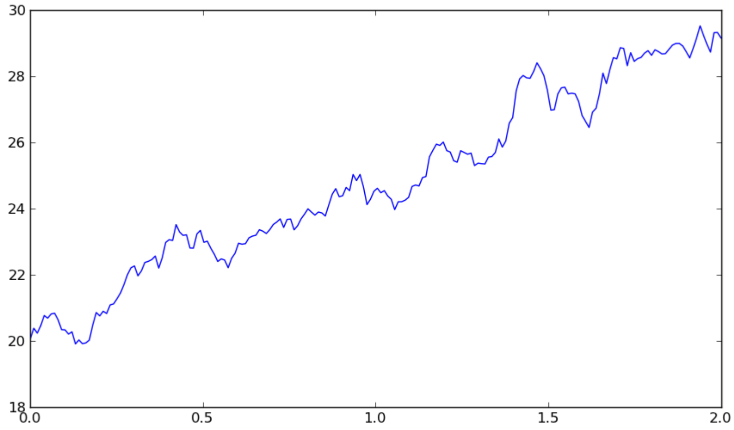
Example: If the interest rate per year is 5%, calculate the effective rate per 7 months and 8 months.

Example: If the interest rate per 10 months is 8%, calculate the effective rate per 7 months, 3 months and 2 months.

Example: General Case: If the interest rate per M months is I%, calculate the effective rate per N months.

Introduction to Force of Interest

Graph A[t]



- From the graph, we can find i_1 and i_2
- But "I" tell the annual increase in the fund, not the increase in any moment of time
- To find the increase in any moment of time A'[t]

Sample Question

1. There are two accounts
   B[t]: Initial principal $100 and money increase by $20 annually
   C[t]: Initial principal $200 and money increase by $20 annually
   You can choose only one account, which account do you choose

Define

$$Force\ of\ Interest[\delta] = \frac{A'[t]}{A[t]}$$

$$\delta[t] = \frac{A'[t]}{A[t]}$$

Solving Different equation:

$$\mathrm{A[t]} = \mathrm{A[0]}e^{\int_0^t \delta[t]dt}$$

$$\mathrm{A}[M] = \mathrm{A[N]}e^{\int_N^M \delta[t]dt}$$

Proof:

Constant force of interest

Fund A[t] grow at a constant force of interest of δ. Find if this fund grows at a Compound interest or Simple Interest and compute the interest rate

constant Force of interest → Effective interest rate

$$e^{\delta} = 1 + I \qquad \delta = \log_e[1 + I]$$

Sample question

1. You are given $\delta[t] = \frac{1}{1+t}$ and A[0]=1 find A[t]

2. Fund A accumulates at a force of interest $\delta[t]$= 10 + 2t.
   Fund B accumulates at a force of interest $\delta[t] = 4t + 3t^2$.
   Equal amounts are invested in each fund at time 0.
   When is the next time the two funds are equal?

2. Bruce deposits 100 into a bank account. His account is credited interest at an annual nominal rate of interest of 4% convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at an annual force of interest of δ.

   After 7.25 years, the value of each account is the same.

   Calculate δ. (0.0396)

Present Value and Future Value

1. The risk-free interest rate is 4%. You are given to choose one of the following option
   A. Receive 100THB now
   B. Receive 103 THB one year later

   What would you choose?

2. The risk-free interest rate is 4%. You are given to choose one of the following option
   A. Receive 50 THB at the end of each year for ever. Starting 1 year from now.
   B. Receive 1000 THB right now

   What would you choose?

- Because of interest, value of money depends on time
- PV: Present Value
- FV: Future Value

Sample Question

1. At the interest rate of 3%. Find the future value of the following cash flow.
   A. +100THB at time 0, find FV at time 3?
   B. - 300THB at time 0, find FV at time 2?

2. At the interest rate of 3%. Find the present value of the following cash flow.
   A. +109.2727THB at time 3
   B. - 318.27THB at time 2

Time(T)	-2.00	-1.00	0.00	1.00	2.00	3.00
A[0]=100,I=3%	฿94.26	฿97.09	฿100.00	฿103.00	฿106.09	฿109.27
A[0]=300,I=3%	฿282.78	฿291.26	฿300.00	฿309.00	฿318.27	฿327.82

Accumulation factor and Discount factor

Accumulation factor

This is factor that is used to send present money into the future

This factor is denoted as (not international symbol)

$$A(a,b)\ when\ a < b$$

And it is used to multiply money at time a to sent it to time b in the future.

Sample questions

1. John will receive a payment of 10 at time 2. Given an interest rate of 5%, calculate the accumulation factor that sent this payment to time 5 and calculate the accumulated value at time 5

2. John will receive a payment of 3 at time 1. Given a force of interest $\delta(t) = \frac{1}{t+5}$. Calculate the accumulation factor that sent this payment to time 10 and calculate the accumulated value at time 10

Discount factor

This is factor that is used to send future money back to present

This factor is denoted as (not international symbol)

$$D(a,b)\ when\ a > b$$

And it is used to multiply money at time a to send it to time b in the past.

Sample questions

1. John will receive a payment of 15 at time 4. Given an interest rate of 5%, calculate the discount factor and discount this payment back to time 0.

2. John will receive a payment of 5 at time 10. Given a force of interest $\delta(t) = \frac{1}{t+3}$. Calculate the discount factor that sent this payment back to time 0 and calculates its present value.

Quiz

1.

Bruce deposits 100 into a bank account. His account is credited interest at an annual nominal rate of interest of 4% convertible semiannually.

At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at an annual force of interest of δ.

After 7.25 years, the value of each account is the same.

Calculate δ.

(A) 0.0388
(B) 0.0392
(C) 0.0396
(D) 0.0404
(E) 0.0414

2.

Eric deposits 100 into a savings account at time 0, which pays interest at an annual nominal rate of i, compounded semiannually.

Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of i.

Eric and Mike earn the same amount of interest during the last 6 months of the 8^{th} year.

Calculate i.

(A) 9.06%
(B) 9.26%
(C) 9.46%
(D) 9.66%
(E) 9.86%

3.

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest for the first 10 years is credited at a nominal discount rate of d compounded quarterly, and thereafter at a nominal interest rate of 6% compounded semiannually. The accumulated balance in the fund at the end of 30 years is 100.

Calculate d.

(A) 4.33%

(B) 4.43%

(C) 4.53%

(D) 4.63%

(E) 4.73%

4.

Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest $\delta_t = \frac{t^2}{100}$, $t > 0$.

The amount of interest earned from time 3 to time 6 is also X.

Calculate X.

(A) 385

(B) 485

(C) 585

(D) 685

(E) 785

5.

David can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n years, and 300 at time $2n$ years

(ii) 600 at time 10 years

At an annual effective interest rate of i, the present values of the two streams are equal.

Given $v^n = 0.76$, calculate i.

(A) 3.5%

(B) 4.0%

(C) 4.5%

(D) 5.0%

(E) 5.5%

6.

Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \le t \le 10$.

Interest is credited at a force of interest $\delta_t = \dfrac{1}{8+t}$.

After time 10, the account is worth 20,000.

Calculate k.

(A) 111

(B) 116

(C) 121

(D) 126

(E) 131