Section 1: Basic probability, conditional probability, independence, combinatoric and permutation

Outline

- Event
- Venn Euler diagram
- Union of events A and B
- Intersect of events A and B
- Mutually exclusive outcomes
- Complement of event
- Subevent (or subset)
- Independent of event A and B
- Conditional probability of event A given B
- Permutation
- Combinatorics
- Permutation with duplicate element
- Quiz

Event

Something that may or may not happen, with probability between 0 and 1 (Inclusive)

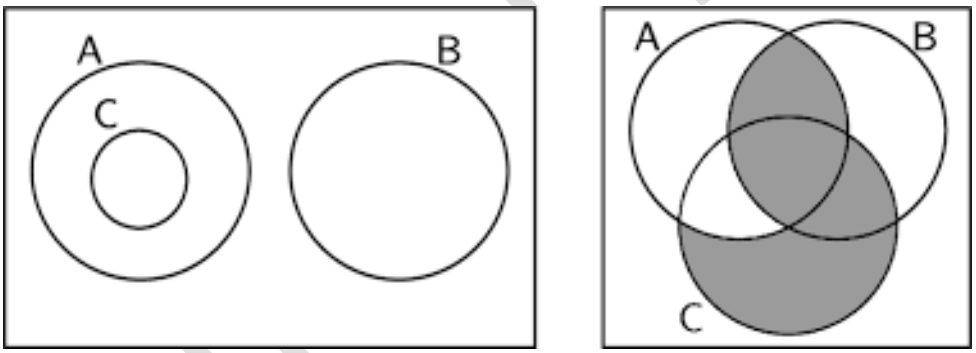
Examples:

- Getting a face with 1 dot when roll a die
- It will rain tomorrow

Let A be any event

P(A) is the probability of event A happening

Venn Euler diagram



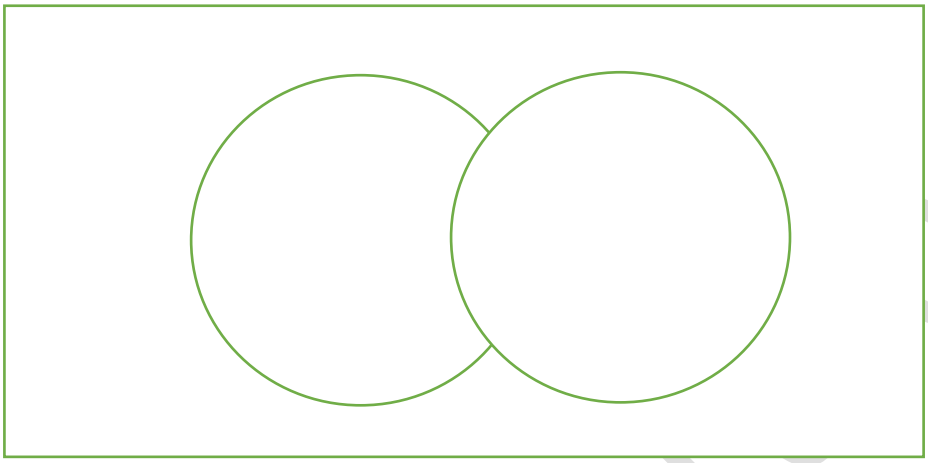
There are two type of venn euler diagram

Type 1: Probability	Type 2: Object
A, B and C are events	A, B and C are categories
Area A = P(A)	Area A = Number in category A
Area B = P(B)	Area B = Number in category B
Area C = P(C)	Area C = Number in category C
Total area = 1	Total area = Total object

Union of events A and B

Symbol: $P(A \cup B)$

Union means: “or”



Sample questions

1. There are 100 people. 30 people play soccer and 40 people play basketball. 40 people don’t play any sport. Calculate the number of people who play soccer or basketball (or both). (60)

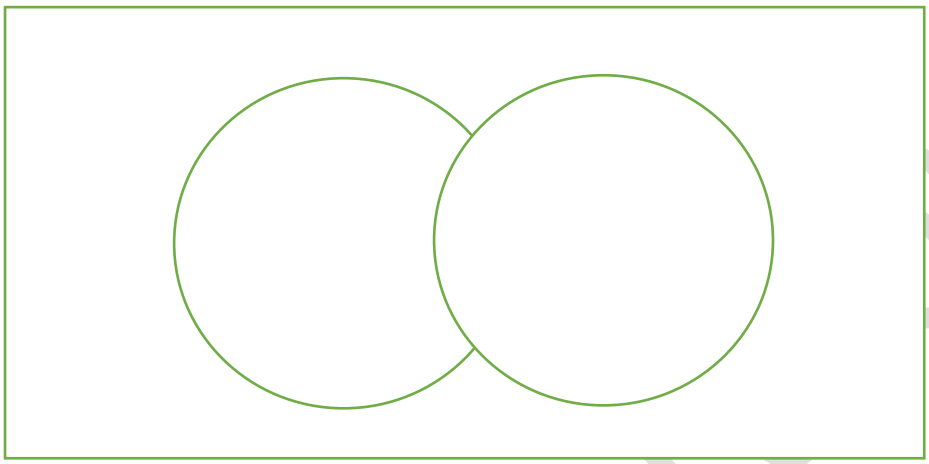
2. There are 100 people.
    - 50 people play soccer or basketball (or both)
    - 60 people play soccer or handball (or both)
    - 55 people play basketball or handball (or both)
    - 35 people play soccer
    - 30 people play basketball
    - 35 people play handball
    - 5 people play all three sports

    Calculate the number of people that play any sport. (70)

Intersect of events A and B

Symbol: $P(A \cap B)$

Intersect means: “and”



Sample questions

1. Toss one coin, if the outcome is head then roll two dice and if the outcome is tail then roll one die.

   Let
   Event A: Outcome of the coin is head
   Event B: Sum of the dice is 2

   Calculate $P(A \cap B)$

2. There are 100 people. 30 people play soccer and 40 people play basketball. 40 people don't play any sport. Calculate the number of people who play soccer and basketball. (10)

Mutually exclusive outcomes

$$Event\ A\ and\ B\ are\ mutually\ exclusive$$
$$\leftrightarrow Event\ A\ and\ B\ can't\ happen\ simultaneously$$

↔ Means “If and on if”

Venn Euler diagram

Sample questions

1. Toss one coin, if the outcome is head then roll two dice and if the outcome is tail then roll one die.

   Let
   Event A: Outcome of the coin is head
   Event B: Sum of the dice is 2

   Are A and B mutually exclusive?

   Let
   Event A: Outcome of the coin is tail
   Event B: Sum of the dice is 7

   Are A and B mutually exclusive?

2. Toss a coin, let
   Event A: The outcome is tail
   Event B: The outcome is head

   Are A and B mutually exclusive outcome?

3. let
   Event A: It will rain tomorrow
   Event B: Stock price will increase tomorrow

   Are A and B mutually exclusive outcome?

Complement of event

Symbol: A'

A' can be described as a set of events that is not in A

Venn Euler diagram

Sample questions

1. Roll a die, let event A be define as face with one dot turns up.
   Explain the complement of event A

2. Each person can either doesn't play sport, play soccer, play basketball or play both
   Let event A be that tom play both sports. Explain the complement of event A

3. There are 100 people in the room. Let event A be that all 100 people are male. Explain the complement of event A

Subevent (or subset)

Symbol: $A \subseteq B$

Read: A is a subset of B

Interpretation: Set A is a subset of B a set if A is contained in B

Venn Euler diagram

Samples question

Determine if event A is a subset of event B or event B is a subset of event A

1. Event A: Tom plays soccer and baseball
   Event B: Tom plays soccer or baseball

2. Event A: Tom is infected by the Coronavirus
   Event B: Tom is dead due to the Coronavirus

3. Event A: Tom misses his 9am class
   Event B: Tom wakes up at 11 am

Independent of event A and B

Important

Description: Event A and B are not associated to each other. A doesn't influence B and B doesn't influence A.

$$\text{A independent from B} \leftrightarrow P(A \cap B) = P(A)xP(B)$$

↔ Means "If and only if"

Venn Euler diagram

Sample questions

1. Throw a die. Let
   Event A: Face with one dot turns up one the first toss
   Event B: Face with one dot turns up one the second toss
   Is A and B independent?

2. There is a loaded coin, with 70% chance of a head turning up and 30% of tail turning up. Toss that coin
   Event A: Head shows up on the first toss
   Event B: Head shows up on the second toss
   Is A and B independent?

Conditional probability of event A given B

"Very important"

Symbol: $P(A \text{ I } B)$

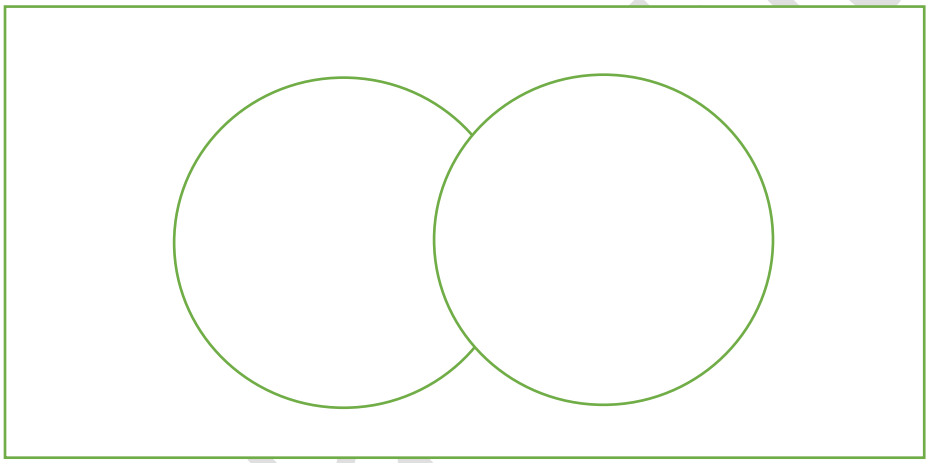
Interpretation: Probability that A happens given that B happens. Usually B will affect A in some way so that $P(A \text{ I } B) \neq P(A)$

Bayes' Theorem

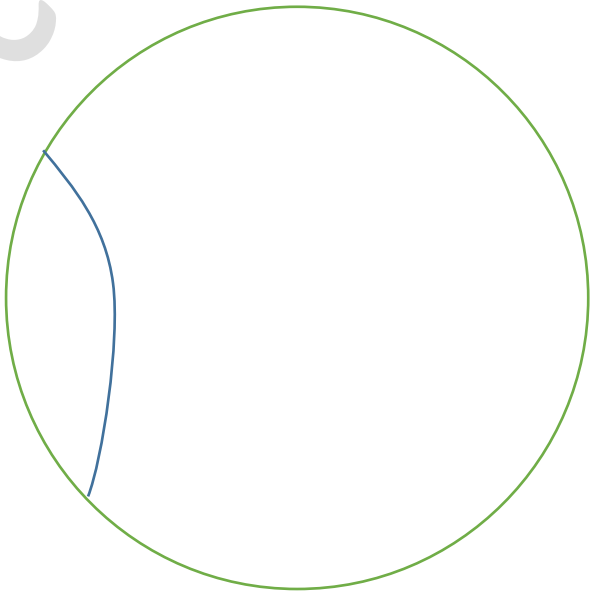
$$P(A \text{ I } B) = \frac{P(A \cap B)}{P(B)}$$

Proof

P(A)=



$P(A \text{ I } B)$



Sample questions

1. Tom must die within 10 years and he can only die at the end of each year. The probability of death in each point is uniformly distributed. Calculate the following:
    a. Find the probability that Tom will die at the end of year 10
    b. Tom survive passes year 9. Find the probability that Tom will die at the end of year 10
    c. Find the probability that Tom will die at the end of year 5
    d. Tom survive passes year 3. Find the probability that Tom will die at the end of year 5

2. There are two dice: dice one with 1,2,3,4,5 and 6 dots on the faces and dice two with 3,3,3,6,6 and 6 dots on the face. Randomly select one die and roll.
    a. Find the probability that the face with 3 dots turn up
    b. Find the probability that the face with 1 dot turn up
    c. Find the probability that the face with 6 dots turn up, given that dice one is selected
    d. Find the probability that the face with 6 dots turn up, given that dice two is selected
    e. Find the probability that dice 1 is selected, given that face with 1 dot turns up
    f. Find the probability that dice 1 is selected, given that face with 3 dot turns up
    g. If face with 3 dots turn up on the first toss, find the probability that face with 3 dots will turn up on the second toss
    h. If face with 1 dot turn up on the first toss, find the probability that face with 1 dot will turn up on the second toss

3. There are two loaded coins. Coin one has a 70% chance of showing a head and 30% tail. Coin two has a 30% chance of showing a head and 70% tail. Randomly choose one coin and toss.
    a. Given that coin one is chosen, find the probability that the outcome is head
    b. Given that coin two is chosen, find the probability that the outcome is tail
    c. Given that the outcome of the first toss is head, find the probability that the chosen coin is coin one
    d. Given that the outcome of the first toss is head, find the probability that the second toss is also head
    e. Given that both first and second toss's outcomes are head, find the probability that the chosen coin is coin one
    f. Given that both first and second toss's outcomes are head, find the probability that the third toss is also head

Law of total probability

Formula

$$P(A) = \sum P(A\ given\ B_n)P(B_n)$$

Sample question

1. Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available and Y supplies 40% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

2. I have three bags that each contain 100 marbles:

   Bag 1 has 75 red and 25 blue marbles.

   Bag 2 has 60 red and 40 blue marbles.

   Bag 3 has 45 red and 55 blue marbles.

   I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Permutation

Symbol: P_r^n

Interpretation: There are n objects (each different from one another). Randomly pick r objects and arrange them into a line. The number of possible different arrangement equal to P_r^n

Sample questions

1. Calculate P_3^5

2. Calculate P_6^{10}

3. Calculate P_2^4

Formula for permutation

$$P_r^n = \frac{n!}{(n-r)!}$$

Proof:

Combinatorics

Symbol: C_r^n

Interpretation: There are n objects (each different from one another). Randomly pick r object (and not arrange them into a line). The number of possible different ways to pick equal to C_r^n

Sample questions

1. Calculate C_3^5

2. Calculate C_2^4

3. Calculate C_1^3

Formula for combinatorics

$$C_r^n = \frac{n!}{(n-r)!\,r!}$$

Proof:

Permutation with duplicate element

There are n objects. In which n_i objects are type i, i=1,2,3,4… and $\sum n_i = n$. The number of distinct permutations is

$$\frac{n!}{n_1! n_2! n_3! \ldots}$$

Proof:

Sample questions

Determine the distinct permutation for the following set

1. $\{A, B, C, C\}$

2. $\{1,1,1,2,2,5,5,5,5,5\}$

3. $\{A, B, C, D, E\}$

4. $\{A, B, C, C, C\}$

Quiz

1. Let $F(x) = \int_0^{x^{\frac{1}{3}}} \sqrt{1+t^4}\, dt$ Find $F'(0)$

(Undefined)

2. Among a large group of patients recovering from shoulder injuries, it is found that 22 % visit both a physical therapist and a chiropractor, whereas l2% visit neither of these. The probability that a patient visits a chiropractor exceeds by 0.14 the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.

(0.48)

3. (SOA) An urn contains l0 balls: 4 red and 6 blue. A second urn contains 16 red balls and an unknown number of blue balls. A single ball is drawn from each urn. The probability that both balls are the same color is 0.44 Calculate the number of blue balls in the second urn.

(4)

4. Two bowls each contain 5 black and 5 white balls. A ball is chosen at random from bowl 1 and put into bowl 2. A ball is then chosen at random from bowl 2 and put into bowl l. Find the probability that bowl 1 still has 5 black and 5 white balls.

(6/11)

5. A box contains 35 gems, of which 10 are real diamonds and 25 are fake diamonds. Gems are randomly taken out of the box, one at a time without replacement. What is the probability that exactly 2 fakes are selected before the second real diamond is selected?

(0.130)

6. A box contains 4 red balls and 6 white balls. A sample of size 3 is drawn without replacement from the box. What is the probability of obtaining 1 red ball and 2 white balls, given that at least 2 of the balls in the sample are white?

(0.75)

7. There are 100 people and 3 sport: soccer, basketball and handball. Each individual that play sport can play up to three sports. Given the following:

- 80 people play sports
- 50 people play soccer
- 40 people play basketball
- 30 people play handball
- 10 people play all 3 sports

Calculate the number of people who play only one sport.

(50)